

# Separation of Variables and Integral Manifolds in One Problem of Motion of Generalized Kowalevski Top

Michael P. Kharlamov and Alexander Y. Savushkin

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## Abstract

In the phase space of the integrable Hamiltonian system with three degrees of freedom used to describe the motion of a Kowalevski-type top in a double constant force field, we point out the four-dimensional invariant manifold. It is shown that this manifold consists of critical motions generating a smooth sheet of the bifurcation diagram, and the induced dynamic system is Hamiltonian with certain subset of points of degeneration of the symplectic structure. We find the transformation separating variables in this system. As a result, the solutions can be represented in terms of elliptic functions of time. The corresponding phase topology is completely described.

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**Key words.** Kowalevski top, double field, invariant manifold

## Introduction

The equations of rotation of a rigid body about a fixed point in a double constant force field have the form

$$\begin{aligned} \mathbf{I} \frac{d\boldsymbol{\omega}}{dt} &= \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{r}_1 \times \boldsymbol{\alpha} + \mathbf{r}_2 \times \boldsymbol{\beta}, \\ \frac{d\boldsymbol{\alpha}}{dt} &= \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \frac{d\boldsymbol{\beta}}{dt} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \end{aligned} \tag{1}$$

where  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are vectors immovable with respect to the body,  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are vectors immovable in the inertial space,  $\mathbf{I}$  is the tensor of inertia at the fixed point  $O$ , and  $\boldsymbol{\omega}$  is the instantaneous angular velocity (all of these objects are expressed via their components relative to certain axes strictly attached to the body).

It is assumed that the vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$  have the origin  $O$ . The points specified by these vectors in the moving space are called the centers of rigging.

System (1) is a Hamiltonian system in the phase space  $P^6$  specified in  $\mathbf{R}^9(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta})$  by geometric integrals;  $P^6$  is diffeomorphic to the tangent bundle  $TSO(3)$ .

In [3], it is proposed to use the problem of motion of a magnetized rigid body in gravitational and magnetic fields and the problem of motion of a rigid body with constant distribution of electric charge in gravitational and electric fields as physical models of Eqs. (1). The results obtained for system (1) in [3] are presented in more details in [4] within the framework of investigation of the Euler equations on Lie algebras.

In the general case  $\mathbf{r}_1 \times \mathbf{r}_2 \neq 0$  and  $\boldsymbol{\alpha} \times \boldsymbol{\beta} \neq 0$ , system (1) without additional restrictions imposed on the parameters, unlike the classical Euler–Poisson equations, is not reducible to a Hamiltonian system with two degrees of freedom and does not have known first integrals on  $P^6$ , except for the energy integral

$$H = \frac{1}{2} \mathbf{I}\boldsymbol{\omega} \cdot \boldsymbol{\omega} - \mathbf{r}_1 \cdot \boldsymbol{\alpha} - \mathbf{r}_2 \cdot \boldsymbol{\beta}.$$

In [3], the following assumptions are introduced for system (1): in the principal axes of the inertia tensor

$$O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \tag{2}$$

the moments of inertia satisfy the conditions  $I_1 = I_2 = 2I_3$  and the vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are parallel to the equatorial plane  $O\mathbf{e}_1\mathbf{e}_2$  and mutually orthogonal. For  $\boldsymbol{\beta} = 0$ , the problem reduces to the Kowalevski

integrable case of rotation of a heavy rigid body [8]. Therefore, for the sake of brevity, the problem proposed in [3] is called the generalized Kowalevski case and the problem with  $\beta = 0$  is called the classical Kowalevski case.

By the appropriate choice of measurement units and axes (2) one can obtain

$$\mathbf{I} = \text{diag}\{2, 2, 1\}; \quad (3)$$

$$\mathbf{r}_1 = \mathbf{e}_1, \quad \mathbf{r}_2 = \mathbf{e}_2. \quad (4)$$

In [3], a new general integral is indicated for the generalized Kowalevski case. In virtue of relations (3) and (4), this integral admits a representation:

$$K = (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2, \quad (5)$$

where  $\omega_i, \alpha_i$ , and  $\beta_i$  ( $i = 1, 2, 3$ ) are the components of the vectors  $\boldsymbol{\omega}, \boldsymbol{\alpha}$ , and  $\boldsymbol{\beta}$  relative to the reference frame (2).

In [10], integral (5) is generalized to the case of motion of a gyrostat in a linear force field by supplementing a body having property (3) with an inner rotor generating a constant moment along the axis of dynamic symmetry. As shown, e.g. in [5], the component of the moment generated by potential forces introduced in [10] can be reduced to the same form as in Eqs. (1) with property (4).

The complete Liouville integrability of the Kowalevski gyrostat in a double force field was proved in [2]. The Lax representation of equations of type (1) (with gyroscopic term in the moment of external forces, as in [10]) containing the spectral parameter was obtained under conditions (3) and (4). The spectral curve of this representation made it possible to find a new first integral which is in involution with the corresponding generalization of integral (5) and turns into the square of the momentum integral for  $\beta = 0$ . Multi-dimensional analogs of the Kowalevski problem were introduced in [2]. It was proposed to solve these problems by the method of finite-band integration. This program was realized in [2] for the classical Kowalevski top and new expressions for the phase variables in the form of special hyper-elliptic functions of time were obtained. The explicit integration of the problem of motion of the Kowalevski top in a double field and its qualitative and topological analysis have not been performed yet (see also a survey in [5]).

The topological analysis of an integrable Hamiltonian system includes the description (in certain terms) of the foliation of its phase space into Liouville tori. In particular, this requires finding all separating cases. These cases correspond to the points of the bifurcation diagram of the integral map and, in the phase space, are formed by the trajectories completely consisting of the points at which the first integrals are not independent.

In a system with three degrees of freedom, two-dimensional Liouville tori are, as a rule, filled with special motions corresponding to a point of the smooth two-dimensional sheet of the bifurcation diagram. The union of these tori over all points of the sheet is an invariant subset of the phase space. In the neighborhood of a point of general position, this subset is a four-dimensional manifold, and the dynamic system induced on this subset must be Hamiltonian with two degrees of freedom (degenerations of various kinds are expected at the boundary of this sheet or at the points of intersection of sheets). Thus, the invariant subsets of maximum dimension formed by the points of dependence of integrals are specified (at least, locally) by systems of two invariant relations of the form

$$f_1 = 0, \quad f_2 = 0. \quad (6)$$

The knowledge of all these systems and the analysis of the dynamics on invariant manifolds specified by these systems is essential to fulfil the topological analysis of the entire problem.

In the generalized Kowalevski case, we know two systems of the form (6). The first one is obtained in [3]. Consider the manifold

$$\{K = 0\} \subset P^6. \quad (7)$$

Due to the structure of function (5), this manifold is specified by two independent equations  $Z_1 = 0$  and  $Z_2 = 0$ . An additional partial integral (Poisson bracket  $\{Z_1, Z_2\}$ ) is indicated. The topological analysis of the induced Hamiltonian system with two degrees of freedom is carried out in [11]. It turns out that the invariant set is a four-dimensional manifold, which is smooth everywhere but the restriction to it of the symplectic structure degenerates on the set of zeros of the additional integral. This case generalizes the first Appelrot class (Delone class) [1] of motions of the classical Kowalevski case.

The second system of the form (6) is obtained in [7]. It is shown that, for  $\beta = 0$ , the corresponding motions transform into so-called *especially remarkable* motions of the second and third Appelrot classes. The present work is devoted to the investigation of the dynamic system on the invariant subset indicated in [7].

First, we make a general remark important for the sequel. The moment of external forces  $\mathbf{r}_1 \times \boldsymbol{\alpha} + \mathbf{r}_2 \times \boldsymbol{\beta}$  appearing in (1) is preserved by the change

$$\begin{pmatrix} \tilde{\mathbf{r}}_1 \\ \tilde{\mathbf{r}}_2 \end{pmatrix} = \Theta \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}, \quad \begin{pmatrix} \tilde{\boldsymbol{\alpha}} \\ \tilde{\boldsymbol{\beta}} \end{pmatrix} = (\Theta^T)^{-1} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}, \quad (8)$$

where  $\Theta$  is an arbitrary non-singular  $2 \times 2$  matrix. Therefore, the *a-priori* assumption made in [3], [2] concerning the orthogonality of the radius vectors of the centers of rigging is redundant (it suffices to require that these centers lie in the equatorial plane of the body). This statement is trivial enough; the possibility of orthogonalization of any pair  $(\mathbf{r}_1, \mathbf{r}_2)$  or  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  is indicated, e.g., in [5]. However, the authors of [5] also indicate that, in general case, the second pair is not orthogonal. Moreover, in [3, 4, 2, 11, 7], the angle between  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  remains arbitrary. This fact makes the corresponding formulas more complicated.

Note that if the pair  $(\mathbf{r}_1, \mathbf{r}_2)$  is made orthonormal, then there remains the arbitrary choice of  $\Theta \in SO(2)$ . Under such transformation, a new pair of radius vectors of the centers of rigging remains orthonormal and can be used as equatorial unit vectors of the principal axes (2) to preserve properties (4). At the same time, by choosing

$$\Theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \tan 2\theta = \frac{2\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\alpha^2 - \beta^2},$$

we get the orthogonal pair  $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ .

Thus, without loss of generality, in addition to relations (4), we can assume that the force fields are orthogonal. This simple fact has not been indicated yet. The elimination of the redundant parameter makes it possible to significantly simplify all subsequent calculations and to obtain results in a symmetric form.

## 1 Invariant Subset and Its Properties

In the sequel, we consider system (1) under assumptions (3) and (4) with the phase space  $P^6$  specified by the formulas

$$\alpha^2 = a^2, \quad \beta^2 = b^2, \quad \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0. \quad (9)$$

The case  $a = b$  is singular. Indeed, as indicated in [10], in this case, there exist a group of symmetries generated by the transformations of the configuration space and, hence, a cyclic integral linear in the angular velocities. Therefore, for the sake of being definite, we set

$$a > b. \quad (10)$$

Denote by  $G$  the general integral of the problem obtained in [2] and represent it in the form

$$G = \frac{1}{4}(g_\alpha^2 + g_\beta^2) + \frac{1}{2}\omega_3 g_\gamma - b^2 \alpha_1 - a^2 \beta_2, \quad (11)$$

where  $g_\alpha$ ,  $g_\beta$ , and  $g_\gamma$  are the scalar products of the kinetic momentum  $\mathbf{I}\boldsymbol{\omega}$  and the vectors  $\boldsymbol{\alpha}$ ,  $\boldsymbol{\beta}$ ,  $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ , respectively.

Introduce the function  $F$  by setting

$$F = (2G - p^2 H)^2 - r^4 K,$$

where the parameters  $p$  and  $r$  are specified as follows:

$$p^2 = a^2 + b^2, \quad r^2 = a^2 - b^2. \quad (12)$$

The latter is well defined due to inequality (10). Obviously,  $F$  is a combined first integral of Eqs. (1).

Note that the zero level of the function  $F$  is specified by one of the conditions

$$2G - p^2 H - r^2 \sqrt{K} = 0; \quad (13)$$

$$2G - p^2H + r^2\sqrt{K} = 0. \quad (14)$$

If  $\beta = 0$ , then these conditions reduce to the equations of the second and third Appelrot classes, respectively [1].

Define the subset  $N \subset P^6$  as the set of critical points of the function  $F$  lying on the level  $F = 0$ .

The set  $N$  is definitely non-empty; it contains, e.g., all points of the form  $\omega_1 = \omega_2 = 0$ ,  $\alpha_1 - \beta_2 = 0$ , and  $\alpha_2 + \beta_1 = 0$ , which are critical for  $K$  and turn the expression  $2G - p^2H$  into zero.

The set  $N$  is invariant under the phase flow of (1) as the set of critical points of the general integral.

The condition  $dF = 0$  means that the differentials of the functions  $H, K$ , and  $G$  are linearly dependent at the points of  $N$ . It immediately implies that the relation

$$(2g - p^2h)^2 - r^4k = 0 \quad (15)$$

for the constants of these integrals is the equation of one of the sheets of the bifurcation diagram (the investigation of this diagram has not been completed yet) of the generalized Kowalevski top.

We use the following complex change of variables [7] (a generalization of the Kowalevski change; see [8]):

$$\begin{aligned} x_1 &= (\alpha_1 - \beta_2) + i(\alpha_2 + \beta_1), & x_2 &= \overline{x_1}, \\ y_1 &= (\alpha_1 + \beta_2) + i(\alpha_2 - \beta_1), & y_2 &= \overline{y_1}, \\ z_1 &= \alpha_3 + i\beta_3, & z_2 &= \overline{z_1}, \\ w_1 &= \omega_1 + i\omega_2, & w_2 &= \overline{w_1}, & w_3 &= \omega_3. \end{aligned} \quad (16)$$

Denote the operation of differentiation with respect to the imaginary time  $it$  by primes and rewrite the equations of motion in terms of the new variables:

$$\begin{aligned} x'_1 &= -x_1w_3 + z_1w_1, & x'_2 &= x_2w_3 - z_2w_2, \\ y'_1 &= -y_1w_3 + z_2w_1, & y'_2 &= y_2w_3 - z_1w_2, \\ 2z'_1 &= x_1w_2 - y_2w_1, & 2z'_2 &= -x_2w_1 + y_1w_2, \\ 2w'_1 &= -(w_1w_3 + z_1), & 2w'_2 &= w_2w_3 + z_2, & 2w'_3 &= y_2 - y_1. \end{aligned} \quad (17)$$

Constraints (9) now take the form

$$\begin{aligned} z_1^2 + x_1y_2 &= r^2, & z_2^2 + x_2y_1 &= r^2, \\ x_1x_2 + y_1y_2 + 2z_1z_2 &= 2p^2. \end{aligned} \quad (18)$$

Further on, instead of integral (11), it is convenient to consider another general integral linearly expressed via  $G$  and  $H$ , namely,

$$M = \frac{1}{r^4}(2G - p^2H). \quad (19)$$

On the level  $F = 0$ , we have

$$K = r^4M^2. \quad (20)$$

In terms of variables (16), rewrite the integrals  $H$ ,  $K$ , and  $M$  as follows:

$$\begin{aligned} H &= w_1w_2 + \frac{1}{2}w_3^2 - \frac{1}{2}(y_1 + y_2), & K &= U_1U_2, \\ M &= -\frac{1}{2r^4}F_1^2 + \frac{1}{2r^2}(U_1 + U_2), \end{aligned}$$

where

$$\begin{aligned} F_1 &= \sqrt{x_1x_2}w_3 - \frac{1}{\sqrt{x_1x_2}}(x_2z_1w_1 + x_1z_2w_2), \\ U_1 &= \frac{x_2}{x_1}(w_1^2 + x_1), & U_2 &= \frac{x_1}{x_2}(w_2^2 + x_2). \end{aligned}$$

Consider the function

$$F_2 = U_1 - U_2.$$

**Proposition 1.1.** *In the domain  $x_1x_2 \neq 0$ , the invariant set  $N$  is specified by the following system of functionally independent equations:*

$$F_1 = 0, \quad F_2 = 0. \quad (21)$$

*Proof.* Represent relation (20) in the form

$$[F_1^2 - r^2(\sqrt{U_1} - \sqrt{U_2})^2][F_1^2 - r^2(\sqrt{U_1} + \sqrt{U_2})^2] = 0, \quad (22)$$

where  $\sqrt{U_1}$  and  $\sqrt{U_2}$  are chosen to be complex conjugates of each other.

On the level  $F = 0$ , the functions  $F_1$ ,  $\sqrt{U_1}$ , and  $\sqrt{U_2}$  are independent everywhere except for the set

$$w_1w_2 = 0, \quad x_1 = x_2. \quad (23)$$

Therefore, the condition that the left-hand side of Eq. (22) has a critical point leads to Eqs. (21). It is clear that points (23) also satisfy these equations. Thus, it remains to notice that the functions  $F_1$  and  $F_2$  are independent on the level  $F = 0$  everywhere in their domain of definition including points (23).  $\square$

The system of invariant relations (21) is obtained in [7] without using the first integrals. In virtue of the above definition and Proposition 1.1, the indicated system describes a certain smooth four-dimensional (non-closed) manifold

$$N^4 = N \cap \{x_1x_2 \neq 0\},$$

and  $N$  is the least invariant subset of  $P^6$  containing  $N^4$ .

**Remark 1.1.** It is easy to see that the invariant set  $N$ , as a whole, is stratified, namely,

$$N = \bigcup_{i=1}^4 N^i, \quad \dim N^i = i, \quad \partial N^i \subset \bigcup_{j=1}^{i-1} N^j.$$

Moreover, in virtue of Proposition 1.1, all  $N^i$  with  $i < 4$  belong to a subset of the phase space specified by the equation

$$x_1x_2 = 0 \quad (24)$$

(e.g.,  $N^1 = \{w_1w_2 = 0, w_3 = 0, x_1x_2 = 0\}$  is diffeomorphic to  $2S^1$ ). Therefore, the existence of singularity (24) in the expressions for  $F_1$  and  $F_2$  is in no case accidental. If, in relations (21), we remove the denominators, then the set of solutions of the obtained system contains the entire four-dimensional manifold (24). This manifold is not everywhere critical for the function  $F$  (however,  $F$  is identically zero on it). In particular, all trajectories starting from this manifold fill a set in  $P^6$ , which is almost everywhere five-dimensional.

The following statement demonstrates that if we restrict ourselves to relations (21), i.e., study the dynamics only on  $N^4$ , then we do not lose any trajectory of the dynamic system on  $N$ .

**Proposition 1.2.** *Set (24) does not contain subsets invariant under the phase flow of system (1).*

To prove this proposition, it is necessary to compute the derivatives of  $x_1x_2$  in virtue of Eqs. (17) up to the fourth order, inclusively, and show that they cannot vanish simultaneously on the set specified by relation (24). It is worth noting that the indicated strong degeneration of this subset also takes place for motions of the heavy Kowalevski top. In that case, condition (24) means that the axis of dynamic symmetry of the top is vertical. Special attention is given to this phenomenon both in the classical papers (see, e.g., [1]) and in recent investigations dealing with the computer animation of motion (see [9]), where one can also find an extensive bibliography of works in this field devoted to the investigation of heavy Kowalevski tops).

**Proposition 1.3.** *The differential 2-form induced on the manifold  $N^4$  by the symplectic structure of the space  $P^6$  providing the Hamiltonian property of Eqs. (1) is non-degenerate everywhere except for the subset specified by the equation  $L = 0$ , where*

$$L = \frac{1}{\sqrt{x_1x_2}} \left[ w_1w_2 + \frac{x_1x_2 + z_1z_2}{2r^2} (U_1 + U_2) \right].$$

*Proof.* The Poisson brackets of the functions on  $\mathbf{R}^9(\omega, \alpha, \beta)$  specifying the indicated symplectic structure are determined according to the following rules [3]:

$$\begin{aligned} \{g_i, g_j\} &= \varepsilon_{ijk} g_k, & \{g_i, \alpha_j\} &= \varepsilon_{ijk} \alpha_k, & \{g_i, \beta_j\} &= \varepsilon_{ijk} \beta_k, \\ \{\alpha_i, \alpha_j\} &= \{\beta_i, \beta_j\} = \{\alpha_i, \beta_j\} = 0, \end{aligned} \quad (25)$$

where  $g_1 = 2\omega_1$ ,  $g_2 = 2\omega_2$ , and  $g_3 = \omega_3$  are the components of the kinetic momentum.

In relations (25), we now pass to variables (16) and compute the Poisson bracket for the functions  $F_1$  and  $F_2$ . In view of relations (21), this gives

$$\{F_1, F_2\} = -r^2 L.$$

The tangent space  $T_q N^4$  is a skew-orthogonal complement of the span of vectors included in the Hamiltonian fields with Hamiltonians  $F_1$  and  $F_2$ . By the Cartan formula (see, e.g., [6], p. 231), the restriction of the symplectic structure to  $T_q N^4$  is non-degenerate provided that  $\{F_1, F_2\}(q) \neq 0$ .  $\square$

**Proposition 1.4.** *The function  $L$  is the first integral of the dynamic system induced on  $N^4$ . Moreover, this integral is in involution with the integral  $M$ .*

*Proof.* As shown in [7], in virtue of (17) we can write

$$F'_1 = \mu_1 F_2, \quad F'_2 = \mu_2 F_1.$$

Here  $\mu_1$  and  $\mu_2$  are functions smooth in the neighborhood of  $N^4$ . In view of these equalities, apply the Jacobi identity to the functions  $H$ ,  $F_1$ , and  $F_2$  and obtain that the double Poisson bracket  $\{H, \{F_1, F_2\}\}$  is a linear combination of the functions  $F_1$  and  $F_2$ . Therefore,  $L' \equiv 0$  on the set specified by relation (21).

It is shown by direct calculation that the following relation is true under conditions (21):

$$L^2 = 2p^2 M^2 + 2HM + 1 \quad (26)$$

and, therefore,

$$L\{L, M\} = M\{H, M\} \equiv 0.$$

It means that  $\{L, M\} = 0$  for  $L \neq 0$ . Hence, by continuity,  $\{L, M\} = 0$  everywhere on  $N^4$ .  $\square$

Thus, in the smooth part  $N^4$  of the invariant subset  $N$  completely specifying the entire dynamics on  $N$ , the equations of motion of the generalized Kowalevski top define the Hamiltonian system with two degrees of freedom with a closed subset of points of degeneration of the symplectic structure nowhere dense in  $N^4$ .

## 2 Analytic Solution

By Proposition 1.4, to integrate the equations of motion in the set  $N$ , we can use the integrals  $M$  and  $L$ . The original general integrals  $H$ ,  $K$ , and  $G$  are expressed via these integrals by using relations (19), (20), and (26).

**Theorem 2.1.** *On an arbitrary integral manifold*

$$J_{m,\ell} = \{M = m, L = \ell\} \subset N, \quad (27)$$

*the equations of motion are separated in the variables*

$$s_1 = \frac{x_1 x_2 + z_1 z_2 + r^2}{2\sqrt{x_1 x_2}}, \quad s_2 = \frac{x_1 x_2 + z_1 z_2 - r^2}{2\sqrt{x_1 x_2}} \quad (28)$$

*and take the form*

$$\begin{aligned} s'_1 &= \sqrt{s_1^2 - a^2} \sqrt{ms_1^2 - \ell s_1 + \frac{1}{4m}(\ell^2 - 1)}, \\ s'_2 &= \sqrt{s_2^2 - b^2} \sqrt{ms_2^2 - \ell s_2 + \frac{1}{4m}(\ell^2 - 1)}. \end{aligned} \quad (29)$$

*Proof.* In virtue of the first equation in (21), the function  $M$  takes the following form on  $N$ :

$$M = \frac{1}{2r^2}(U_1 + U_2).$$

In virtue of the second equation in (21), we get  $U_1 = U_2$ . Therefore, the integral equation  $M = m$  yields

$$U_1 = r^2 m \quad \text{and} \quad U_2 = r^2 m. \quad (30)$$

Determine  $w_3$  from the first equation in (21) and  $w_1$  and  $w_2$  from Eqs. (30). We obtain

$$w_3 = \frac{z_1 w_1}{x_1} + \frac{z_2 w_2}{x_2}, \quad w_2 = \sqrt{\frac{x_2}{x_1}} R_1, \quad w_1 = \sqrt{\frac{x_1}{x_2}} R_2, \quad (31)$$

where

$$R_1 = \sqrt{r^2 m - x_1} \quad \text{and} \quad R_2 = \sqrt{r^2 m - x_2}. \quad (32)$$

Substituting these quantities in the equation of the integral  $L$ , we obtain

$$m(x^2 + z^2) - \ell x + \sqrt{r^4 m^2 - 2r^2 m x \cos \sigma + x^2} = 0. \quad (33)$$

The variables  $x, z$ , and  $\sigma$  are defined as follows

$$x^2 = x_1 x_2, \quad z^2 = z_1 z_2, \quad x_1 + x_2 = 2x \cos \sigma, \quad (34)$$

and the radical in (33) corresponds to  $w_1 w_2$ , and therefore is non-negative. The other radicals used above, including  $R_1$  and  $R_2$ , are algebraic.

Equation (33) now yields

$$R_1 R_2 = \ell x - m(x^2 + z^2),$$

$$R_1^2 + R_2^2 = \frac{1}{r^2 m} \{ [\ell x - m(x^2 + z^2)]^2 - x^2 \} + r^2 m,$$

Introducing the polynomial

$$\Phi(s) = 4ms^2 - 4\ell s + \frac{1}{m}(\ell^2 - 1),$$

we can write in terms of variables (28)

$$R_1 + R_2 = \frac{r}{s_1 - s_2} \sqrt{\Phi(s_2)} \quad \text{and} \quad R_1 - R_2 = \frac{r}{s_1 - s_2} \sqrt{\Phi(s_1)}. \quad (35)$$

Using constraints (18) and relations (34), we obtain

$$(z_1 \pm z_2)^2 = \frac{1}{r^2} [(x^2 + z^2 \pm r^2)^2 - 2x^2(p^2 \pm r^2)].$$

Hence, in terms of variables (28),

$$z_1 + z_2 = \frac{2r}{s_1 - s_2} \sqrt{s_1^2 - a^2}, \quad z_1 - z_2 = \frac{2r}{s_1 - s_2} \sqrt{s_2^2 - b^2}. \quad (36)$$

We now differentiate relations (28) in virtue of system (17). In view of (31), we get

$$s'_1 = \frac{r^2}{4x^2} (z_1 + z_2)(R_1 - R_2) \quad \text{and} \quad s'_2 = \frac{r^2}{4x^2} (z_1 - z_2)(R_1 + R_2).$$

Substituting expressions (35) and (36) in these equalities, we arrive at system (29).  $\square$

**Remark 2.1.** It is clear that the deduced equations can be integrated in elliptic functions of time. By using the standard procedure, the solutions are expressed in terms of Jacobi functions. Their specific form depends on the location of the roots of the polynomials under the radicals on the right-hand sides. The bifurcation solutions of systems of this type correspond to stationary points of one of the equations, i.e., to the cases for which the polynomial

$$(s^2 - a^2)(s^2 - b^2)\Phi(s) \quad (37)$$

possesses a multiple root.

For dimension reasons, the original phase variables on manifold (27) are expressed via  $s_1$  and  $s_2$ , though, in general, these expressions might be multi-valued functions. We now show that the latter have a fairly simple algebraic form.

Introduce the following notation:

$$S_1 = \sqrt{s_1^2 - a^2}, \quad \varphi_1 = \sqrt{-\Phi(s_1)}, \quad (38)$$

$$S_2 = \sqrt{b^2 - s_2^2}, \quad \varphi_2 = \sqrt{\Phi(s_2)};$$

$$\psi = 4ms_1s_2 - 2\ell(s_1 + s_2) + \frac{1}{m}(\ell^2 - 1). \quad (39)$$

**Theorem 2.2.** *On the common level of the first integrals (27), by using notation (38), (39), the phase variables of the generalized Kowalevski case can be expressed, in terms of variables (28), as follows:*

$$\begin{aligned} \alpha_1 &= \frac{1}{2(s_1 - s_2)^2} [(s_1s_2 - a^2)\psi + S_1S_2\varphi_1\varphi_2], \\ \alpha_2 &= \frac{1}{2(s_1 - s_2)^2} [(s_1s_2 - a^2)\varphi_1\varphi_2 - S_1S_2\psi], \\ \beta_1 &= -\frac{1}{2(s_1 - s_2)^2} [(s_1s_2 - b^2)\varphi_1\varphi_2 - S_1S_2\psi], \\ \beta_2 &= \frac{1}{2(s_1 - s_2)^2} [(s_1s_2 - b^2)\psi + S_1S_2\varphi_1\varphi_2], \\ \alpha_3 &= \frac{r}{s_1 - s_2} S_1, \quad \beta_3 = \frac{r}{s_1 - s_2} S_2, \\ \omega_1 &= \frac{r}{2(s_1 - s_2)} (\ell - 2ms_1)\varphi_2, \quad \omega_2 = \frac{r}{2(s_1 - s_2)} (\ell - 2ms_2)\varphi_1, \\ \omega_3 &= \frac{1}{s_1 - s_2} (S_2\varphi_1 - S_1\varphi_2). \end{aligned} \quad (40)$$

*Proof.* By using notation (12), we represent the compatibility conditions of constraints (18) in the variables  $x$  and  $z$  as follows:

$$x^2 + z^2 + r^2 \geq 2a|x|, \quad |x^2 + z^2 - r^2| \leq 2b|x|,$$

whence we get the *natural* ranges for variables (28):

$$s_1^2 \geq a^2 \quad \text{and} \quad s_2^2 \leq b^2. \quad (41)$$

Hence, rewriting Eqs. (29) in the real form, we conclude that, for given  $m$  and  $\ell$ , the domain of possible motions in the plane  $(s_1, s_2)$  is determined, along with inequalities (41), by the inequalities

$$\Phi(s_1) \leq 0 \quad \text{and} \quad \Phi(s_2) \geq 0. \quad (42)$$

Thus, in particular, all values (38) are real on the trajectories of the analyzed system. The expressions for the complex variables  $x_1$ ,  $x_2$ ,  $z_1$ ,  $z_2$ ,  $w_1$ ,  $w_2$ , and  $w_3$  in terms of  $s_1$  and  $s_2$  are obtained by application, in sequence, of relations (35) with regard for (32), then (36), and, finally, (31). After this, the variables  $y_1$  and  $y_2$  are determined from the first two relations in (18). By the change of variables inverse to (16), we arrive at the required dependencies (40).  $\square$

Note that the value  $s_1 = \infty$  is ordinary for the first equation in (29) (because the degree of the polynomial under the radical on the right-hand side is even). Moreover, if this value belongs to the domain of possible motions, then it is reached during a finite period of time. Similarly, relations (40) also do not have singularities in this case. This can be proved by the change of variables  $s_1 \mapsto 1/s_1$ . Thus, in particular, we have deduced analytic expressions for all cases in which the trajectories pass the set specified by relation (24). It means that we have constructed the complete analytic solution of the problem on the invariant set  $N$ .



### 3 Phase Topology

In the regular case, the integral manifold  $J_{m,\ell}$  consists of two-dimensional Liouville tori. The cases when they topologically rearrange generate the bifurcation diagram of the system on  $N$ . It is natural to study this diagram in the plane of constants of the used integrals, i.e., as the set of critical values of the map

$$J = M \times L : N \rightarrow \mathbf{R}^2. \quad (43)$$

**Theorem 3.1.** *The bifurcation diagram of map (43) is a part of the system of straight lines*

$$\ell = -2ma \pm 1, \ell = 2ma \pm 1, \ell = -2mb \pm 1, \ell = 2mb \pm 1, \quad (44)$$

and the coordinate axes of the plane  $(\ell, m)$  lying in the half-plane  $\ell \geq 0$  and specified by the conditions of existence of real solutions

$$\begin{aligned} \ell &\geq \max(2ma - 1, -2mb + 1), & m > 0; \\ \ell &\leq -2mb + 1, & m < 0; \\ \ell &= 1, & m = 0. \end{aligned} \quad (45)$$

*Proof.* According to Remark 2.1, the diagram contains the discriminant set of polynomial (37) formed by straight lines (44) (in the part corresponding to the existence of motions).

The points of the coordinate axes in the plane  $(m, \ell)$  belonging to  $J(N)$  must be included in the diagram; indeed, it can be shown that the values  $m = 0$  and  $\ell = 0$  are attained, in particular, on the subsets  $N^i$  ( $i < 4$ ), where  $N$  fails to be smooth (see Remark 1.1). The analytical foundation for this inclusion is as follows.

In Eqs. (29), we can pass to the limit as  $m \rightarrow 0$ . As a result, by using relations (26), we conclude that  $|\ell| \rightarrow 1$  and  $(\ell^2 - 1)/2m \rightarrow h$ . At the same time, the degree of the polynomials under the radicals decreases to three; the form of solutions changes. Moreover, it is clear that  $K$  equals zero on the set  $N \cap \{M = 0\}$ , i.e., the corresponding motions also belong to the class (7). [It is worth noting that, as shown in [11], the restriction of symplectic structure to manifold (7) degenerates just at the points of corresponding trajectories.] Therefore, the value  $m = 0$  should be regarded as corresponding to bifurcation.

On the other hand, the integral surface  $J_{m,\ell}$  is preserved by the inversion

$$(\alpha_3, \beta_3, \omega_3) \mapsto (-\alpha_3, -\beta_3, -\omega_3).$$

In relations (40), this inversion is realized either by changing the sign of the radicals  $S_1$  and  $S_2$  or by the substitution  $(\ell, s_1, s_2) \mapsto (-\ell, -s_1, -s_2)$ . This means that  $J_{m,\ell}$  and  $J_{m,-\ell}$  are the same subset of the phase space. Therefore, we need to restrict ourselves to the values of  $\ell$  of the same sign (to be definite, we choose non-negative values); the axis  $\ell = 0$  becomes the outer boundary of the domain of existence of motions in the plane of constants of the integrals. In virtue of Eq. (33),  $\ell$  can equal zero only for negative values of  $m$ .

Thus, in addition to (44), the diagram should also be supplemented with the point  $\{m = 0, \ell = 1\}$  and the semi-axis

$$\{\ell = 0, m < 0\}. \quad (46)$$

By analyzing the compatibility of conditions (41) and (42), we determine the actual region of existence of motions in the form (45).  $\square$

In Fig. 1, the domains with numbers 1–9 defined by the diagram in the plane  $(m, \ell)$  correspond to different types of the integral surfaces (27). The motion is impossible in the shaded region.

To determine the number of tori for the regular manifold, we note that relations (40) give a one-valued dependence of the phase variables on two collections of quantities

$$(s_1, S_1, \varphi_1) \quad \text{and} \quad (s_2, S_2, \varphi_2). \quad (47)$$

In this case, the signs of the radicals in (38) on each  $J_{m,\ell}$  are arbitrary. However, along the trajectory, some radicals turn to zero and then change the sign. This means that two points that only differ by the sign of such radical lie in the same connected component of  $J_{m,\ell}$ . Therefore, the number of connected components of the regular integral manifold is equal to  $2^n$ , where  $n$  is the number of quantities (38) non-zero along the trajectory. The value of  $n$  is determined according to the location of roots of polynomial (37) and does not exceed 2.

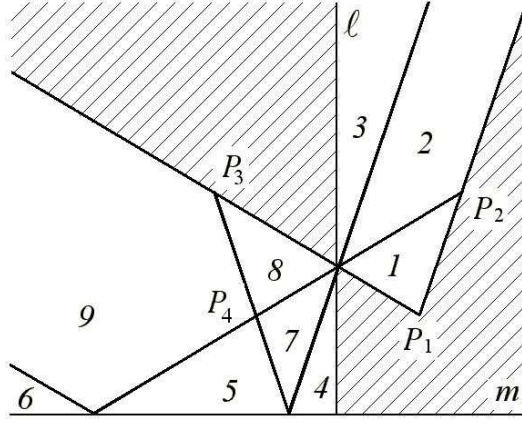


Figure 1: Bifurcation diagram and the domains of existence of motions

**Proposition 3.1.** *Assume that the analyzed domains are numbered as indicated in Fig. 1. Then the integral manifolds  $J_{m,\ell}$  can be described as follows: a)  $\mathbf{T}^2$ , in domains 1 and 8, b)  $2\mathbf{T}^2$ , in domains 2, 7, and 9, and c)  $4\mathbf{T}^2$ , in domains 3–6.*

To determine the type of critical integral surfaces, we note that, in each three-dimensional space of collections (47), equalities (38) specify a pair of cylinders (elliptic or hyperbolic) with mutually orthogonal generatrices. For the points of the straight lines (44), a pair of cylinders corresponding to one of the variables  $s_1$  or  $s_2$  has a tangency point. Hence, the line of their intersection is the eight curve  $S^1 \vee S^1$ . Thus, on segments of the straight lines (44) bounded by the points of lines intersection and internal for domain (45), the integral surface consists of the components homeomorphic to the product  $S^1 \times (S^1 \vee S^1)$ . Crossing such segment, we observe one of bifurcations  $\mathbf{T}^2 \rightarrow 2\mathbf{T}^2$  typical for systems with two degrees of freedom. The number of connected components of the form  $S^1 \times (S^1 \vee S^1)$  in the critical  $J_{m,\ell}$  is determined by the number of tori in the adjacent domains. Actually, the critical periodic trajectories (the traces of centers of the eight curve) are motions in which one of variables  $s_1$  or  $s_2$  remains constant and equal to the multiple root of the corresponding polynomial under the radical. In this case, either  $S_1 \equiv 0$  and  $\varphi_1 \equiv 0$  or  $S_2 \equiv 0$  and  $\varphi_2 \equiv 0$ . Hence, it follows from relations (40) that  $\omega_2 = \omega_3 \equiv 0$  in the first case and  $\omega_1 = \omega_3 \equiv 0$  in the second case. The body performs pendulum motions in which the radius vector of one of the centers of rigging is permanently directed along the corresponding force field. In approaching the outer boundary of domain (45) with the exception of the half-line (46), the tori degenerate into circles (periodic solutions of the indicated pendulum type) and the surfaces  $S^1 \times (S^1 \vee S^1)$  degenerate into eight curves.

It is clear that the critical single-frequency motions do not appear in the half-line (46). The corresponding bifurcation in the segments adjacent to domains 5 and 6 is characterized by the fact that the number of connected components of  $J_{m,0}$  is half as large as at the close regular point of the plane  $(m, \ell)$ . These are so-called minimal tori. The transition from domain 4 to a segment of the boundary set (46) is not accompanied by the decrease in the number of components of  $J_{m,\ell}$  and all cycles homotopic to a certain marked cycle are folded so that each component covers the limiting component twice. In a sufficiently smooth case (e.g., in the case when  $L$  is a Bott integral on the corresponding smooth level of the integral  $M$ ), a Klein bottle should be obtained as a result (see, e.g., [6]). However, according to the explicit equations (40), this is not true in our case. Most likely, this phenomenon is connected with the degeneration of the induced symplectic structure.

Finally, consider the nodes denoted by  $P_1$ – $P_4$  in Fig. 1. For these values of the constants of integrals, each surface  $J_{m,\ell}$  contains one singular point. These points correspond to the equilibria of the body in which both centers of rigging lie on the corresponding axes of force fields and, hence, the moment of forces is equal to zero. One of these points is stable: at  $P_1$ , the integral surface consists of a single point. The other three points are unstable. As indicated above, at the nodes  $P_2$  and  $P_3$  the integral surface is homeomorphic to an eight curve. At the node  $P_4$  the integral surface can be described as follows. Take a rectangle and identify its vertices with one point; it then can be filled with trajectories double-asymptotic to the singular point. The boundary of this set is a bunch of four circles. This boundary represents two pairs of pendulum motions; each pair is asymptotic to the highest position of one of the two centers of rigging. Take four copies of the

obtained set and attach the boundary of each to the same bunch of four circles.

All indicated phenomena are readily established by analyzing relations (40) and the mutual location of the cylinders formed in spaces (47).

## Conclusions

In the present work, we perform the complete investigation of motions of the generalized Kowalevski top playing the role of critical motions for the entire problem and generating bifurcations of three-dimensional Liouville tori along paths crossing the sheet specified by Eq. (15) of the bifurcation diagram  $\Sigma \subset \mathbf{R}^3$  of the general integrals of the problem. Inequalities (45) are used to deduce the equations of the boundary of a part of this sheet corresponding to the existence of actual critical motions, i.e., contained in  $\Sigma$ .

Consider relation (22). It plays the role of the equation of the entire integral surface in the phase space  $P^6$  for the collection of constants of integrals satisfying relation (15). It then follows, similar to the case of the heavy Kowalevski top (the second and third Appelrot classes), that the straight line  $\{k = 0, 2g = p^2 h\}$  splits sheet (15) into two classes. In the first class specified by relation (13) and corresponding to the first non-negatively definite factor in (22), the obtained integral manifolds, being critical for the original system, exhaust the entire corresponding integral surface in  $P^6$  as the limit of a concentric family of three-dimensional tori and are, in this sense, stable in  $P^6$ . In the second class specified by relation (14) and corresponding to the second (hyperbolic) factor in (22), all obtained critical surfaces in  $P^6$  are hyperbolically unstable: on the same level of the first three integrals, one can find trajectories consisting of regular points and double-asymptotic to the corresponding two-dimensional tori of the system on the investigated invariant set.

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